Chapter
Sequences and Series

Topic-1: Arithmetic Progression

1 MCQs with One Correct Answer

1. Let $b_i > 1$ for $i = 1, 2, ..., 101$. Suppose $\log_e b_1, \log_e b_2, ..., \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then [Adv. 2016]
(a) $s > t$ and $a_{101} > b_{101}$ (b) $s > t$ and $a_{101} < b_{101}$
(c) $s < t$ and $a_{101} > b_{101}$ (d) $s < t$ and $a_{101} < b_{101}$

2. In the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is [2009]
(a) $\frac{n(4n^2-1)c^2}{6}$ (b) $\frac{n(4n^2+1)c^2}{3}$
(c) $\frac{n(4n^2-1)c^2}{3}$ (d) $\frac{n(4n^2+1)c^2}{6}$

3. If the sum of the first $2n$ terms of the A.P. 2, 5, 8, …, is equal to the sum of the first n terms of the A.P. 57, 59, 61, …, then n equals [2001S]
(a) 10 (b) 12 (c) 11 (d) 13

2 Integer Value Answer/Non-Negative Integer

4. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is [Adv. 2015]

5. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = [Adv. 2013]

6. Let $a_1, a_2, a_3 .....a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is [2011]

7. Let S_k, $k = 1, 2, ... , 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k|$ is [2010]

8. Let $a_1, a_2, a_3 ...., a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, ...11$. if $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ... + a_{11}}{11}$ is equal to [2010]

3 Numeric/New Stem Based Questions

9. Let $l_1, l_2, ..., l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, ..., w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2, where $d_1 d_2 = 10$. For each $i = 1, 2, ..., 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ________. [Adv. 2022]

10. Let AP$(a;d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If AP$(1;3) \cap$ AP$(2;5) \cap$ AP$(3;7)$ = AP$(a;d)$ then $a + d$ equals ____. [Adv. 2019]

11. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ____. [Adv. 2018]

12. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? [Adv. 2018]

4 Fill in the Blanks

13. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A =$ .... and $B =$ .... **[1997 - 2 Marks]**

14. The sum of integers from 1 to 100 that are divisible by 2 or 5 is .... **[1984 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

15. Let $a_1, a_2, a_3, ...$ be an arithmetic progression with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, ...$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE ? **[Adv. 2022]**
(a) $T_{20} = 1604$ (b) $\sum_{k=1}^{20} T_k = 10510$
(c) $T_{30} = 3454$ (d) $\sum_{k=1}^{30} T_k = 35610$

16. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) **[Adv. 2013]**
(a) 1056 (b) 1088 (c) 1120 (d) 1332

17. Let T_r be the r^{th} term of an A.P., for $r = 1, 2, 3, ...$ If for some positive integers m, n we have
$T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals **[1998 - 2 Marks]**
(a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$
(c) 1 (d) 0

8 Comprehension/Passage Based Questions

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r - 1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, ...$

18. The sum $V_1 + V_2 + ... + V_n$ is **[2007 -4 marks]**
(a) $\frac{1}{12}n(n+1)(3n^2 - n + 1)$
(b) $\frac{1}{12}n(n+1)(3n^2 + n + 2)$
(c) $\frac{1}{2}n(2n^2 - n + 1)$
(d) $\frac{1}{3}(2n^3 - 2n + 3)$

19. T_r is always
(a) an odd number (b) an even number
(c) a prime number (d) a composite number

20. Which one of the following is a correct statement ?
(a) $Q_1, Q_2, Q_3, ...$ are in A.P. with common difference 5
(b) $Q_1, Q_2, Q_3, ...$ are in A.P. with common difference 6
(c) $Q_1, Q_2, Q_3, ...$ are in A.P. with common difference 11
(d) $Q_1 = Q_2 = Q_3 = ...$

10 Subjective Problems

21. The fourth power of the common difference of an arithmatic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. **[2000 - 4 Marks]**

22. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. **[1996 - 3 Marks]**

23. If $\log_3 2$, $\log_3(2^x - 5)$, and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x. **[1990 - 4 Marks]**

24. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°, Find the number of sides of the polygon. **[1980]**

Topic-2: Geometric Progression

1 MCQs with One Correct Answer

1. In the quadratic equation $ax^2+bx+c=0$, $\Delta=b^2-4ac$ and $\alpha+\beta$, $\alpha^2+\beta^2$, $\alpha^3+\beta^3$, are in G.P. where α, β are the root of $ax^2+bx+c=0$, then [2005S]
   (a) $\Delta \neq 0$ (b) $b\Delta=0$ (c) $c\Delta=0$ (d) $\Delta=0$
2. An infinite G.P. has first term 'x' and sum '5', then x belongs to [2004S]
   (a) $x<-10$ (b) $-10<x<0$
   (c) $0<x<10$ (d) $x>10$
3. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a<b<c$ and $a+b+c=\frac{3}{2}$, then the value of a is [2002S]
   (a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$
4. Let α, β be the roots of $x^2-x+p=0$ and γ, δ be the roots of $x^2-4x+q=0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q respectively, are [2001S]
   (a) $-2,-32$ (b) $-2, 3$ (c) $-6, 3$ (d) $-6,-32$
5. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then [2000S]
   (a) $a=\frac{4}{7}, r=\frac{3}{7}$ (b) $a=2, r=\frac{3}{8}$
   (c) $a=\frac{3}{2}, r=\frac{1}{2}$ (d) $a=3, r=\frac{1}{4}$
6. Sum of the first n terms of the series
   $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+$.............. is equal to [1988 - 2 Marks]
   (a) 2^n-n-1 (b) $1-2^{-n}$
   (c) $n+2^{-n}-1$ (d) 2^n+1
7. If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in — [1985 - 2 Marks]
   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these
8. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is [1983 - 1 Mark]
   (a) $\frac{2355}{1001}$ (b) $\frac{2379}{997}$
   (c) $\frac{2355}{999}$ (d) none of these
9. The third term of a geometric progression is 4. The product of the first five terms is [1982 - 2 Marks]
   (a) 4^3 (b) 4^5
   (c) 4^4 (d) none of these

2 Integer Value Answer/ Non-Negative Integer

10. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1=b_1=c$, then the number of all possible values of c, for which the equality
    $$2(a_1+a_2+\ldots+a_n)=b_1+b_2+\ldots+b_n$$
    holds for some positive integer n, is _____ [Adv. 2020]

6 MCQs with One or More than One Correct Answer

11. Let n be an odd integer. If $\sin n\theta=\sum_{r=0}^{n} b_r \sin^r\theta$, for every value of θ, then [1998 - 2 Marks]
    (a) $b_0=1, b_1=3$ (b) $b_0=0, b_1=n$
    (c) $b_0=-1, b_1=n$ (d) $b_0=0, b_1=n^2-3n+3$
12. For $0<\phi<\pi/2$, if
    $$x=\sum_{n=0}^{\infty}\cos^{2n}\phi,\ y=\sum_{n=0}^{\infty}\sin^{2n}\phi,\ z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$$
    then: [1993 - 2 Marks]
    (a) $xyz=xz+y$
    (b) $xyz=xy+z$
    (c) $xyz=x+y+z$
    (d) $xyz=yz+x$

10 Subjective Problems

13. If $a_n=\frac{3}{4}-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots..(-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n=1-a_n$, then find the least natural number n_0 such that $b_n>a_n\ \forall\ n\geq n_0$. [2006 - 6M]
14. Let a, b, c, d be real numbers in G.P. If u, v, w, satisfy the system of equations [1999 - 10 Marks]
    $$u+2v+3w=6$$
    $$4u+5v+6w=12$$
    $$6u+9v=4$$
    then show that the roots of the equation
    $$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+[(b-c)^2+(c-a)^2+(d-b)^2]\,x+u+v+w=0$$
    and $20x^2+10(a-d)^2x-9=0$ are reciprocals of each other.

15. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n+1}$ respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \ldots + S_{2n-1}^2$ **[1991 - 4 Marks]**

16. Find three numbers a, b, c, between 2 and 18 such that
(i) their sum is 25
(ii) the numbers $2, a, b$ are consecutive terms of an A.P. and
(iii) the numbers $b, c, 18$ are consecutive terms of a G.P. **[1983 - 2 Marks]**

17. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms ? If it exits, how many such progressions are possible ? **[1982 - 3 Marks]**

Topic-3: Harmonic Progression, Relation Between A. M., G. M. and H. M. of two Positive Numbers

1 MCQs with One Correct Answer

1. Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[2012]**
(a) 22 (b) 23 (c) 24 (d) 25

2. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are **[2001S]**
(a) NOT in A.P./G.P./H.P. (b) in A.P.
(c) in G.P. (d) in H.P.

3. The harmonic mean of the roots of the equation $\left(5+\sqrt{2}\right)x^2 - \left(4+\sqrt{5}\right)x + 8 + 2\sqrt{5} = 0$ is **[1999 - 2 Marks]**
(a) 2 (b) 4 (c) 6 (d) 8

4. Let $a_1, a_2, \ldots a_{10}$ be in A, P, and $h_1, h_2, \ldots h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is **[1999 - 2 Marks]**
(a) 2 (b) 3 (c) 5 (d) 6

5. If $ln(a+c)$, $ln(a-c)$, $ln(a-2b+c)$ are in A.P., then **[1994]**
(a) a, b, c are in A.P. (b) a^2, b^2, c^2 are in A.P.
(c) a, b, c are in G.P. (d) a, b, c are in H.P.

2 Integer Value Answer/ Non-Negative Integer

6. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ____ **[Adv. 2020]**

4 Fill in the Blanks

7. Let the harmonic mean and geometric mean of two positive numbers be the ratio 4 : 5. Then the two number are in the ratio .... **[1992 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

8. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then **[2008]**
(a) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (b) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$
(c) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ (d) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

9. If $x > 1, y > 1, z > 1$ are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in **[1998 - 2 Marks]**
(a) A.P. (b) H.P.
(c) G.P. (d) None of these

10. If the first and the $(2n-1)$st terms of an A.P., a G.P. and an H.P. are equal and their n-th terms are a, b and c respectively, then **[1988 - 2 Marks]**
(a) $a = b = c$ (b) $a \geq b \geq c$
(c) $a + c = b$ (d) $ac - b^2 = 0$.

8 Comprehension/Passage Based Questions

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, Let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively. **[2007 -4 marks]**

11. Which one of the following statements is correct ?
(a) $G_1 > G_2 > G_3 > \ldots$
(b) $G_1 < G_2 < G_3 < \ldots$
(c) $G_1 = G_2 = G_3 = \ldots$
(d) $G_1 < G_3 < G_5 < \ldots$ and $G_2 > G_4 > G_6 > \ldots$

12. Which one of the following statements is correct ?
(a) $A_1 > A_2 > A_3 > \ldots$
(b) $A_1 < A_2 < A_3 < \ldots$
(c) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 < \ldots$
(d) $A_1 < A_3 < A_5 < \ldots$ and $A_2 > A_4 > A_6 > \ldots$

13. Which one of the following statements is correct ?
(a) $H_1 > H_2 > H_3 > \ldots$
(b) $H_1 < H_2 < H_3 < \ldots$
(c) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 < \ldots$
(d) $H_1 < H_3 < H_6 < \ldots$ and $H_2 > H_4 > H_6 > \ldots$

9 Assertion and Reason/Statement Type Questions

14. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

STATEMENT - 1 : The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. and
STATEMENT - 2 : The numbers b_1, b_2, b_3, b_4 are in H.P. **[2008]**
(a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is a correct explanation for STATEMENT - 1
(b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 is **NOT** a correct explanation for STATEMENT - 1
(c) STATEMENT - 1 is True, STATEMENT - 2 is False
(d) STATEMENT - 1 is False, STATEMENT - 2 is True

10 Subjective Problems

15. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either $a=b=c$ or $a, b, -\frac{c}{2}$ form a G.P. **[2003 - 4 Marks]**

16. Let a, b be positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that $\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}=\frac{(2a+b)(a+2b)}{9ab}$. **[2002 - 5 Marks]**

17. Let $a_1, a_2, ..., a_n$ be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of $a_1, a_2, ..., a_n$. Find an expression for the geometric mean of $G_1, G_2, ..., G_n$ in terms of $A_1, A_2, ..., A_n, H_1, H_2, ..., H_n$. **[2001 - 5 Marks]**

18. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$. **[1991 - 4 Marks]**

19. If $a>0, b>0$ and $c>0$, prove that
$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$ **[1984 - 2 Marks]**

20. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation.
$2A+G^2=27$
Find the two numbers. **[1979]**

Topic-4: Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences

2 Integer Value Answer/ Non-Negative Integer

1. Let $7\overbrace{5\cdots5}^{r}7$ denote the (r + 2) digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S=77+757+7557+\cdots+7\overbrace{5\cdots5}^{98}7$. If $S=\frac{7\overbrace{5\cdots5}^{99}7+m}{n}$, where m and n are natural numbers less than 3000, then the value of m + n is **[Adv. 2023]**

4 Fill in the Blanks

2. For any odd integer $n\geq 1, n^3-(n-1)^3+...+(-1)^{n-1}1^3=$ .... **[1996 - 1 Mark]**

3. The sum of the first n terms of the series $1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+$......... is $n(n+1)^2/2$, when n is even. When n is odd, the sum is .... **[1988 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

4. Let α and β be the roots of $x^2-x-1=0$, with α > β. For all positive integers n, define
$a_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}, n\geq 1$
$b_1=1$ and $b_n=a_{n-1}+a_{n+1}, n\geq 2$
Then which of the following options is/are correct ? **[Adv. 2019]**
(a) $\sum_{n=1}^{\infty}\frac{a_n}{10^n}=\frac{10}{89}$
(b) $b_n=\alpha^n+\beta^n$ for all $n\geq 1$
(c) $a_1+a_2+a_3+.....a_n=a_{n+2}-1$ for all $n\geq 1$
(d) $\sum_{n=1}^{\infty}\frac{b_n}{10^n}=\frac{8}{89}$

5. For a positive integer n, let
$a(n)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....\frac{1}{(2^n)-1}$. Then **[1999 - 3 Marks]**
(a) $a(100)\leq 100$ (b) $a(100)>100$
(c) $a(200)\leq 100$ (d) $a(200)>100$

10 Subjective Problems

6. Find the sum of the series :
$\sum_{r=0}^{n}(-1)^r\,{}^nC_r[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}$..... up to m terms] **[1985 - 5 Marks]**

7. If n is a natural number such that
$n=p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}......p_k^{\alpha_k}$ and $p_1, p_2, ..., p_k$ are distinct primes, then show that $\ln n\geq k\ln 2$ **[1984 - 2 Marks]**

Answer Key

Topic-1 : Arithmetic Progression

1. (b)	**2.** (c)	**3.** (c)	**4.** (9)	**5.** (5)	**6.** (9)	**7.** (3)	**8.** (0)	**9.** (18900)
10. (157)	**11.** (3748)	**12.** (6)	**13.** (–3, 77)	**14.** (3050)	**15.** (b, c)	**16.** (a, d)	**17.** (c)	**18.** (b)
19. (d)	**20.** (b)							

Topic-2 : Geometric Progression

1. (c)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (c)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (1)
11. (b)	**12.** (b, c)								

Topic-3 : Harmonic Progression, Relation Between A. M., G. M. and H.M. of two Positive Numbers

1. (d)	**2.** (d)	**3.** (b)	**4.** (d)	**5.** (d)	**6.** (8)	**7.** (4 : 1 or 1 : 4)	**8.** (b, d)	**9.** (b)
10. (a, b, d)	**11.** (c)	**12.** (a)	**13.** (b)	**14.** (c)				

Topic-4 : Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences

1. (1219) **2.** $\frac{1}{4}(n+1)^2(2n-1)$ **3.** $\frac{n^2(n+1)}{2}$ **4.** (a,b,c) **5.** (a, d)

Hints & Solutions

Topic-1: Arithmetic Progression

1. **(b)** $\log_e b_1, \log_e b_2, ----, \log_e b_{101}$ are in A.P.

$\Rightarrow$ $b_1, b_2, --- , b_{101}$ are in G.P.

Also $a_1, a_2, --- , a_{101}$ are in A.P., where $a_1 = b_1$ are $a_{51} = b_{51}$.

$\therefore$ $b_2, b_3, ---, b_{50}$ are GM's and $a_2, a_3, ---, a_{50}$ are AM's between b_1 and b_{51}.

$\because$ GM < AM $\Rightarrow b_2 < a_2, b_3 < a_3, ---, b_{50} < a_{50}$

$\because$ $b_1 + b_2 + ---- + b_{51} < a_1 + a_2 + ---- + a_{51}$

$\Rightarrow$ $t < s$

Also a_1, a_{51}, a_{101} are in AP and b_1, b_{51}, b_{101} are in GP

$\because$ $a_1 = b_1$ and $a_{51} = b_{51}$, $\therefore$ $b^{101} > a_{101}$

2. **(c)** Given : For an A.P, $S_n = cn^2$

Then $T_n = S_n - S_{n-1} = cn^2 - c(n-1)^2$

$= (2n-1)c$

$\therefore$ Sum of squares of n terms of this A.P.

$$= \sum T_n^2 == \sum (2n-1)^2 .c^2$$

$$= c^2\left[4\sum n^2 - 4\sum n + n\right]$$

$$= c^2\left[\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right]$$

$$= c^2 n\left[\frac{2(2n^2+3n+1) - 6(n+1) + 3}{3}\right]$$

$$= c^2 n\left[\frac{4n^2-1}{3}\right] = \frac{n(4n^2-1)c^2}{3}$$

3. **(c)** Given $2 + 5 + 8 + ... 2n$ terms $= 57 + 59 + 61 + ...$ n terms

$$\Rightarrow \frac{2n}{2}[4 + (2n-1)\,3] = \frac{n}{2}[114 + (n-1)\,2]$$

$$\Rightarrow 6n + 1 = n + 56 \Rightarrow 5n = 55 \Rightarrow n = 11$$

4. **(9)** $$\frac{S_7}{S_{11}} = \frac{6}{11} \Rightarrow \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \Rightarrow a = 9d$$

$a_7 = a + 6d = 15d$

Given $130 < a_7 < 140$

$\Rightarrow$ $130 < 15d < 140 \Rightarrow d = 9$

[Since d is a natural number because all terms are natural numbers.]

5. **(5)** Let k, $k + 1$ be removed from pack.

$\therefore (1 + 2 + 3 + ... + n) - (k + k + 1) = 1224$

$$\frac{n(n+1)}{2} - 2k = 1225 \Rightarrow k = \frac{n(n+1) - 2450}{4}$$

for $n = 50$, $k = 25$, $\therefore$ $k - 20 = 5$

6. **(9)** $$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}[2\times 3 + (5n-1)d]}{\frac{n}{2}[6 + (n-1)d]} \qquad [\because m = 5n]$$

$$= \frac{5[(6-d) + 5nd]}{(6-d) + nd]}$$, which will be independent of n

if $d = 6$ or $d = 0$

For a proper A.P., we take $d = 6$

$\therefore$ $a_2 = a_1 + d = 3 + 6 = 9$

7. **(3)** We know that $S_\infty = \dfrac{a}{1-r}$

$$\therefore \quad S_k = \begin{cases} \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k}}, & k \neq 1 \\ 0, & k = 1 \\ \dfrac{1}{(k-1)!}, & k \geq 2 \end{cases}$$

Now $$\sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = \sum_{k=2}^{100} |(k^2 - 3k + 1)| \frac{1}{(k-1)!}$$

$$= |-1| + \sum_{k=3}^{100} \frac{(k^2-1) + 1 - 3(k-1) - 2}{(k-1)!},$$

since $k^2 - 3k + 1 > 0 \quad \forall\, k \geq 3$

$$= 1 + \sum_{k=3}^{100}\left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right)$$

$$= 1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + ...$$

$$.... + \left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99!}\right)$$

$$= 3 - \frac{1}{98!} - \frac{1}{99!} = 3 - \frac{9900}{100!} - \frac{100}{100!} = 3 - \frac{10000}{100!} = 3 - \frac{(100)^2}{100!}$$

$$\Rightarrow \frac{100^2}{100!} + \sum_{k=1}^{100}\left|(k^2 - 3k + 1)S_k\right| = 3.$$

8. **(0)** Given : $a_k = 2a_{k-1} - a_{k-2}$

$$\Rightarrow \frac{a_{k-2} + a_k}{2} = a_{k-1}, 3 \leq k \leq 11$$

$\Rightarrow a_1, a_2, a_3, ..., a_{11}$ are in AP.

If a is the first term and D the commen difference then

$$a_1^2 + a_2^2 + ... + a_{11}^2 = 990$$

$$\Rightarrow 11a^2 + d^2\left(1^2 + 2^2 + ... + 10^2\right) + 2ad\left(1 + 2 + ... + 10\right)$$

$= 990$

$$\Rightarrow 11a^2 + \frac{10\times 11\times 21}{6}d^2 + 2ad\times\frac{10\times 11}{2} = 990$$

$\Rightarrow a^2+35d^2+10d=90$

Since $a=a_1=15$

$\therefore 35d^2+150d+135=0 \Rightarrow 7d^2+30d+27=0$

$\Rightarrow (d+3)(7d+9)=0 \Rightarrow d=-3 \text{ or } -9/7$

then $a_2=15-3=12$ or $15-\frac{9}{7}=\frac{96}{7}>\frac{27}{2}$

$\therefore d \neq -9/7$

Hence $\frac{a_1+a_2+...+a_{11}}{11}=\frac{\frac{11}{2}[2\times15+10(-3)]}{11}=0$

9. **(18900)** For A.P. $l_1, l_2, ... l_{100}$

Let $T_1=a$ and common difference $=d_1$

and similarly now for A.P. $w_1, w_2, ... w_{100}$

$T_1=b$ and common difference $=d_2$

$A_{51}-A_{50}=l_{51}w_{51}-l_{50}w_{50}$

$=(a+50d_1)(b+50d_2)-(a+49d_1)(b+49d_2)$

$=50bd_1+50ad_2+2500d_1d_2-49ad_2-49bd_1-2401d_1d_2$

$=bd_1+ad_2+99d_1d_2=1000$

$\Rightarrow \quad bd_1+ad_2=1000-990=10 \quad$ ...(i) (As $d_1d_2=10$)

$\therefore \quad A_{100}-A_{90}=l_{100}w_{100}-l_{90}w_{90}$

$=(a+99d_1)(b+99d_2)-(a+89d_1)(b+89d_2)$

$=99bd_1+99ad_2+99^2d_1d_2-89bd_1-89ad_2-89^2d_1d_2$

$=10(bd_1+ad_2)+1880d_1d_2$

$\Rightarrow \quad 10(10)+1880(10)=18900$

10. **(157)**

AP (1, 3) : 1, 4, 7, 10, 13 ....

AP (2, 5) : 2, 7, 12, 17, 22 ....

AP (3, 7) : 3, 10, 17, 24, 31 ....

For AP (1, 3) $\cap$ AP (2, 5) $\cap$ AP (3, 7)

first term will be the minimum common value of a term.

$\therefore$ we need to find that minimum number which.

when divided by 7 leaves remainder 3 $\rightarrow (7m+3)$

and when divided by 5 leaves remainder 2 $\rightarrow (5p+2)$

and when divided by 3 leaves remainder 1 $\rightarrow (3q+1)$

By hit and trial 52 is such number $(7\times7+3)$

$\therefore$ first term 'a' of intersection AP = 52

Also common difference 'd' of intersection AP

= LCM (7, 5, 3) = 105

$\therefore a+d=52+105=157$

11. **(3748)** The given sequences upto 2018 terms are

1, 6, 11, 16, ..., 10086

and 9, 16, 23, ..., 14128

The common terms are

16, 15, 86, ... upto n terms, where $T_n \le 10086$

$\Rightarrow \quad 16+(n-1)35 \le 10086$

$\Rightarrow \quad 35n-19 \le 10086$

$\Rightarrow \quad n \le \frac{10105}{35}=288.7$

$\therefore \quad n=288$

$\therefore \quad n(X\cup Y)=n(X)+n(Y)-n(X\cap Y)$

$=2018+2018-288=3748$

12. **(6)** Let the sides be a – d, a, a + d where d is positive. Using Pythagoras theorem,

$(a+d)^2=(a-d)^2+a^2 \Rightarrow a=4d$

$\therefore$ Sides are 3d, 4d, 5d

Area = 24 $\Rightarrow \frac{1}{2}\times3d\times4d=24 \Rightarrow d^2=4 \Rightarrow d=2$

$\therefore$ Smallest side = 3d = 6.

13. Since p and q are the roots of $x^2-2x+A=0$

$\therefore \quad p+q=2, pq=A$

Also r and s are the roots of equation $x^2-18x+B=0$

$\therefore \quad r+s=18, rs=B$

Since, p, q, r, s are in A.P.

Therefore, let $p=a-3d, q=a-d, r=a+d$ and $s=a+3d$.

As $p<q<r<s$, we have $d>0$

Now, $2=p+q=a-3d+a-d=2a-4d$

$\Rightarrow \quad a-2d=1$ ....(i)

Also $18=r+s=a+d+a+3d$

$\Rightarrow \quad 18=2a+4d \quad \Rightarrow \quad 9=a+2d.$ ....(ii)

On subtracting (i) from (ii), we get

$\Rightarrow \quad 8=4d \Rightarrow 2=d$

On putting in (ii) we obtain a = 5

$\therefore \quad p=a-3d=5-6=-1, \quad q=a-d=5-2=3$

$r=a+d=5+2=7, \quad s=a+3d=5+6=11$

$\Rightarrow \quad A=pq=-3$ and $B=rs=77$.

14. The sum of integers from 1 to 100 that are divisible by 2 or 5 = (sum of integers from 1 to 100 divisible by 2) + (sum of integers from 1 to 100 divisible by 5) – (sum of integers from 1 to 100 divisible by 10)

$=(2+4+6+...+100)+(5+10+15+...+100)$
$-(10+20+...+100)$

$=\frac{50}{2}[2\times2+49\times2]+\frac{20}{2}[2\times5+19\times5]$

$-\frac{10}{2}[2\times10+9\times10]=2550+1050-550=3050$

15. **(b, c)** Given, $a_1=7, d=8$

Hence, $a_n=7+(n-1)8$ and $T_1=3$

Also $T_{n+1}=T_n+a_n$

$T_n=T_{n-1}+a_{n-1}$

$\vdots$

$T_2=T_1+a_1$

$\therefore \quad T_{n+1}=(T_{n-1}+a_{n-1})+a_n$

$=T_{n-2}+a_{n-2}+a_{n-1}+a_n$

$\vdots$

$\Rightarrow \quad T_{n+1}=T_1+a_1+a_2+...+a_n$

$\Rightarrow \quad T_{n+1}=T_1+\frac{n}{2}[2(7)+(n-1)8]$

$\Rightarrow \quad T_{n+1}=3+n(4n+3)$ ...(i)

Hence, for $n=19$; $T_{20}=3+(19)(79)=1504$

For n = 29; $T_{30}=3+(29)(119)=3454 \rightarrow$ (C)

$\sum_{k=1}^{20}T_k=3+\sum_{k=2}^{20}T_k=3+\sum_{k=1}^{19}(3+4n^2+3n)$

$=3+3(19)+\frac{3(19)(20)}{2}+\frac{4(19)(20)(39)}{6}$

$=3+10507=10510 \rightarrow$ (b)

And similarly $\sum_{k=1}^{30}T_k=3+\sum_{k=1}^{29}(4n^2+3n+3)=35615$

16. **(a, d)** $S=-1^2-2^2+3^2+4^2-5^2-6^2+....$

$=(3^2+7^2+11^2+...)+(4^2+8^2+12^2+...)$
$-(1^2+5^2+9^2+...)-(2^2+6^2+10^2+...)$

$=\sum_{r=1}^{n}(4r-1)^2+\sum_{r=1}^{n}(4r)^2-\sum_{r=1}^{n}(4r-3)^2-\sum_{r=1}^{n}(4r-2)^2$

$=\left[\sum_{r=1}^{n}(4r-1)^2-(4r-3)^2\right]+4\left[\sum_{r=1}^{n}(2r)^2-(2r-1)^2\right]$

$=8\sum_{r=1}^{n}(2r-1)+4\sum_{r=1}^{n}(4r-1)$

$=8\left[2\frac{n(n+1)}{2}-n\right]+4\left[4\frac{n(n+1)}{2}-n\right]$

$= 8n^2 + 8n^2 + 4n = 16n^2 + 4n$

For $n = 8$, $16n^2 + 4n = 1056$

and for $n = 9$, $16n^2 + 4n = 1332$

17. (c) $T_m = a + (m-1)\,d = 1/n$

and $t_n = a + (n-1)d = 1/m$

$\Rightarrow (m-n)\,d = 1/n - 1/m = (m-n)/mn \Rightarrow d = 1/mn$

$\therefore \quad a = \dfrac{1}{mn}$

Now, $t_{mn} = a + (mn-1)\,d = \dfrac{1}{mn} + (mn-1)\dfrac{1}{mn}$

$= \dfrac{1}{mn} + 1 - \dfrac{1}{mn} = 1$

18. (b) $V_1 + V_2 + ... + V_n = \sum_{r=1}^{n} V_r = \sum_{r=1}^{n}\left(r^3 - \dfrac{r^2}{2} + \dfrac{r}{2}\right)$

$= \Sigma n^3 - \dfrac{\Sigma n^2}{2} + \dfrac{\Sigma n}{2}$

$= \dfrac{n^2(n+1)^2}{4} - \dfrac{n(n+1)\,(2n+1)}{12} + \dfrac{n(n+1)}{4}$

$= \dfrac{n(n+1)}{4}\left[n(n+1) - \dfrac{2n+1}{3} + 1\right]$

$= \dfrac{n(n+1)\,(3n^2+n+2)}{12}$

19. (d) $T_r = V_{r+1} - V_r - 2$

$= \left[(r+1)^3 - \dfrac{(r+1)^2}{2} + \dfrac{r+1}{2}\right] - \left[r^3 - \dfrac{r^2}{2} + \dfrac{r}{2}\right] - 2$

$= 3r^2 + 2r + 1 = (r+1)(3r-1)$

For each r, T_r has two different factors other than 1 and itself.

$\therefore$ T_r is always a composite number.

20. (b) $Q_{r+1} - Q_r = T_{r+2} - T_{r+1} - (T_{r+1} - T_r)$

$= T_{r+2} - 2T_{r+1} + T_r$

$= (r+3)(3r+5) - 2(r+2)(3r+2) + (r+1)(3r-1)$

$\because Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6$ (constant)

$\therefore Q_1, Q_2, Q_3, ...$ are in AP with common difference 6.

21. Let $a - 3d, a - d, a + d$ and $a + 3d$ be any four consecutive terms of an A.P. with common difference 2d.

Since, terms of A.P. are integers, $2d$ is also an integer.

Now, $(2d)^4 + (a-3d)(a-d)(a+d)(a+3d)$

$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2) = (a^2 - 5d^2)^2$

$\because a^2 - 5d^2 = a^2 - 9d^2 + 4d^2$

$= (a-3d)(a+3d) + (2d)^2 =$ some integer

Hence, the resulting sum is the square of an integer.

22. Since x_1, x_2, x_3 are in A.P., hence we can suppose $x_1 = a - d, x_2 = a$ and $x_3 = a + d$ and x_1, x_2, x_3 are the roots of $x^3 - x^2 + \beta x + \gamma = 0$

Since x_1, x_2, x_3 i.e., $(a-d), a, (a+d)$ are the roots of $x^3 - x^2 + \beta x + \gamma = 0$

$\therefore \quad a - d + a + a + d = 1$ ....(i)

$= (a-d)\,a + a\,(a+d) + (a-d)(a+d) = \beta$ ....(ii)

and $(a-d)\,a\,(a+d) = -\gamma$ ....(iii)

From (i), $3a = 1 \Rightarrow a = 1/3$

From (ii), $3a^2 - d^2 = \beta$

$\Rightarrow 3(1/3)^2 - d^2 = \beta \Rightarrow 1/3 - \beta = d^2$

$\because \quad d^2 \geq 0 \;\forall\; d \in R$

$\therefore \quad \dfrac{1}{3} - \beta \geq 0$

$\Rightarrow \beta \leq \dfrac{1}{3} \Rightarrow \beta \in (-\infty, 1/3]$

From (iii), $a(a^2 - d^2) = -\gamma$

$\Rightarrow \dfrac{1}{3}\left(\dfrac{1}{9} - d^2\right) = -\gamma \Rightarrow \dfrac{1}{27} - \dfrac{1}{3}d^2 = -\gamma$

$\Rightarrow \gamma + \dfrac{1}{27} = \dfrac{1}{3}d^2 \Rightarrow \gamma + \dfrac{1}{27} \geq 0$

$\Rightarrow \gamma \geq -\dfrac{1}{27} \Rightarrow \gamma \in \left[-\dfrac{1}{27}, \infty\right)$

$\therefore \quad \beta \in (-\infty, 1/3)$ and $\gamma \in [-1/27, \infty]$

23. Given : $\log_3 2, \log_3(2^x - 5), \log_3(2^x - 7/2)$ are in A.P.

$\therefore \quad 2\log_3(2^x - 5) = \log_3 2 + \log_3(2^x - 7/2)$

$\Rightarrow (2^x - 5)^2 = 2\left(2^x - \dfrac{7}{2}\right)$

$\Rightarrow (2^x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 12.2^x + 32 = 0$

Let $2^x = y$, then we get,

$y^2 - 12y + 32 = 0 \Rightarrow (y-4)(y-8) = 0$

$\Rightarrow y = 4$ or $8 \Rightarrow 2^x = 2^2$ or $2^3 \Rightarrow x = 2$ or 3

But for $\log_3(2^x - 5)$ and $\log_3(2^x - 7/2)$ to be defined

$2^x - 5 > 0$ and $2^x - 7/2 > 0$

$\Rightarrow 2^x > 5$ and $2^x > 7/2$

$\therefore 2^x > 5$

$\Rightarrow x \neq 2$ and therefore $x = 3$.

24. Let there be n sides in the polygon.

$\therefore$ Sum of all n interior angles of polygon $= (n-2) \times 180°$

Since the angles are in A.P. with the smallest angle 120° and common difference 5°.

$\therefore$ Sum of all interior angles of polygon

$= \dfrac{n}{2}[2\times 120 + (n-1)\times 5]$

$\therefore \quad \dfrac{n}{2}[2\times 120 + (n-1)\times 5] = (n-2)\times 180$

$\Rightarrow \dfrac{n}{2}[5n + 235] = (n-2)\times 180$

$\Rightarrow 5n^2 + 235n = 360n - 720$

$\Rightarrow n^2 - 25n + 144 = 0 \therefore n = 16, 9$

Also if $n = 16$, then 16th angle $= 120 + 15 \times 5 = 195° > 180°$, which is not possible. $\therefore \quad n = 9$.

Topic-2: Geometric Progression

1. (c) In the quadratic equation $ax^2 + bx + c = 0$

$\Delta = b^2 - 4ac$ and $\alpha + \beta = -\dfrac{b}{a}, \alpha\beta = \dfrac{c}{a}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \dfrac{b^2}{a^2} - \dfrac{2c}{a} = \dfrac{b^2 - 2ac}{a^2}$

and $\alpha^3 + \beta^3 = -\dfrac{b^3}{a^3} - \dfrac{3c}{a}\left(-\dfrac{b}{a}\right) = -\left(\dfrac{b^3 - 3abc}{a^3}\right)$

Since $\alpha + \beta, \alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in G.P.

$\therefore \quad -\dfrac{b}{a}, \dfrac{b^2 - 2ac}{a^2}, -\dfrac{(b^3 - 3\,abc)}{a^3}$ are in G.P.

$\Rightarrow \left(\dfrac{b^2 - 2\,ac}{a^2}\right)^2 = \dfrac{b}{a}\left(\dfrac{b^3 - 3\,abc}{a^3}\right)$

$\Rightarrow b^4 + 4a^2c^2 - 4ab^2c = b^4 - 3ab^2c$

$\Rightarrow 4a^2c^2 - ab^2c = 0 \Rightarrow ac\Delta = 0$

$\Rightarrow c\Delta = 0$ ($\because$ In quadratic Eq. $a \neq 0$)

2. (c) $\frac{x}{1-r} = 5 \Rightarrow r = 1 - \frac{x}{5}$

Since G.P. contains infinite terms

$\therefore -1 < r < 1$

$\Rightarrow -1 < 1 - \frac{x}{5} < 1 \Rightarrow -2 < -\frac{x}{5} < 0$

$\Rightarrow -10 < x < 0 \Rightarrow 0 < \frac{x}{5} < 2$

$\therefore 0 < x < 10.$

3. (d) Since a, b, c are in A.P.

$\therefore 2b = a + c$

But given $a + b + c = 3/2 \Rightarrow b = 1/2$ and then $a + c = 1$

Also a^2, b^2, c^2, are in G.P. $\Rightarrow b^4 = a^2c^2$

$\Rightarrow b^2 = \pm ac \Rightarrow ac = \frac{1}{4}$ or $-\frac{1}{4}$

and $a + c = 1$ ....(i)

Considering $a + c = 1$ and $ac = 1/4$

$(a-c)^2 = 1 - 1 = 0 \Rightarrow a = c$ but $a \neq c$

as given that $a < b < c$

$\therefore a + c = 1$ and $ac = -1/4$

$\Rightarrow (a-c)^2 = 1 + 1 = 2 \Rightarrow a - c = \pm\sqrt{2}$

but $a < c \Rightarrow a - c = -\sqrt{2}$ ....(ii)

Solving (i) and (ii), we get $a = \frac{1}{2} - \frac{1}{\sqrt{2}}$

4. (a) α, β are the roots of $x^2 - x + p = 0$

$\therefore \alpha + \beta = 1$ ....(i)

$\alpha\beta = p$ ....(ii)

γ, δ are the roots of $x^2 - 4x + q = 0$

$\therefore \gamma + \delta = 4$ ....(iii)

$\gamma\delta = q$ ....(iv)

$\alpha, \beta, \gamma, \delta$ are in G.P.

$\therefore$ Let $\alpha = a;\ \beta = ar,\ \gamma = ar^2,\ \delta = ar^3$.

Substituting these values in equations (i), (ii), (iii) and (iv), we get

$a + ar = 1$ ....(v)

$a^2r = p$ ....(vi)

$ar^2 + ar^3 = 4$ ....(vii)

$a^2r^5 = q$ ....(viii)

On dividing (vii) by (v), we get

$\frac{ar^2(1+r)}{a(1+r)} = \frac{4}{1} \Rightarrow r^2 = 4 \Rightarrow r = 2, -2$

From (v), $a = \frac{1}{1+r} = \frac{1}{1+2}$ or $\frac{1}{1-2} = \frac{1}{3}$ or -1

Since p is an integer (given), r is also an integer (2 or -2).

From (vi), $a \neq \frac{1}{3}$. Hence $a = -1$ and $r = -2$.

$\therefore p = (-1)^2 \times (-2) = -2$

$q = (-1)^2 \times (-2)^5 = -32$

5. (d) Sum = 4 and second term = 3/4,

$\Rightarrow \frac{a}{1-r} = 4$ and $ar = 3/4 \Rightarrow r = \frac{3}{4a}$

$\therefore \frac{a}{1 - \frac{3}{4a}} = 4 \Rightarrow \frac{4a^2}{4a-3} = 4$

$\Rightarrow a^2 - 4a + 3 = 0 \quad \therefore a = 1$ or 3

When $a = 1$, $r = 3/4$ and when $a = 3$, $r = 1/4$

6. (c) Sum of first n terms $= \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + ... + n$th term

$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) .... + n$th term

$= (1 + 1 + 1 + ... + n\text{th term}) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... + \frac{1}{2^n}\right)$

$= n - \left[\frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}\right] = n - 1 + 2^{-n}$

7. (a) If a, b, c are in G.P., then

$b^2 = ac$ ....(i)

Now $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root

Let it be α, then $a\alpha^2 + 2b\alpha + c = 0$

$d\alpha^2 + 2e\alpha + f = 0$

$\Rightarrow \frac{\alpha^2}{2(bf - ec)} = \frac{\alpha}{cd - af} = \frac{1}{2(ae - bd)}$

$\Rightarrow \alpha^2 = \frac{bf - ce}{ae - bd};\ \alpha = \frac{cd - af}{2(ae - bd)}$

Putting the value of α, in $\alpha^2 = \frac{bf - ce}{ae - bd}$, we get

$\frac{(cd - af)^2}{4(ae - bd)^2} = \frac{bf - ce}{ae - bd}$

$\Rightarrow (cd - af)^2 = 4(ae - bd)(bf - ce)$

On dividing both sides by a^2c^2, we get

$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{bd}{ac}\right)\left(\frac{bf}{ac} - \frac{e}{a}\right)$

$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{d}{b}\right)\left(\frac{f}{b} - \frac{e}{a}\right)$ [using eq. (i)]

$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} - \frac{2df}{ac} = \frac{4ef}{cb} - \frac{4e^2}{ac} - \frac{4df}{b^2} + \frac{4de}{ab}$

$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} + \frac{4e^2}{b^2} + 2\frac{d}{a}.\frac{f}{c} - 4\frac{e}{b}.\frac{f}{c} - 4.\frac{d}{a}.\frac{e}{b} = 0$ [using eq.(i)]

$\Rightarrow \left(\frac{d}{a} + \frac{f}{c} - 2\frac{e}{b}\right)^2 = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$

$\therefore \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

8. (c) $2.\overline{357} = 2 + [.357 + 0.000357 + ...\infty]$

$= 2 + \left[\frac{357}{10^3} + \frac{357}{10^6} + ...\infty\right] = 2 + \frac{\frac{357}{10^3}}{1 - \frac{1}{10^3}}$

$= 2 + \frac{357}{999} = \frac{2355}{999}$

[Sum of a G.P. with infinite term $= \frac{a}{1-r}$]

9. (b) Given : $ar^2 = 4$

Now product of first five terms of a G.P

$= a.\ ar.\ ar^2 .ar^3 .ar^4 = a^5 r^{10} = (ar^2)^5 = 4^5.$

10. (1) It is given that

$2(a_1 + a_2 + ... + a_n) = b_1 + b_2 + ... + b_n$

$\Rightarrow 2\times\frac{n}{2}\left(2c+(n-2)x_2\right)=c\left(\frac{2^n-1}{2-1}\right)$ $[\because a_1=c, b_1=c]$

$\Rightarrow c\left(2^n-1-2n\right)=2n^2-2n$

$\Rightarrow c=\frac{2n^2-2n}{2^n-1-2n}\in\mathbb{N}$

So, $2n^2-2n\geq 2^n-1-2n$

$\Rightarrow 2n^2+1\geq 2^n \Rightarrow n<7$

$\because c\in\mathbb{N}\Rightarrow c>0\Rightarrow n>2$

$\Rightarrow$ n can be 3, 4, 5 or 6.

Checking c against these values of n

When n = 3, c = 14

When n = 4, $c=\frac{24}{7}$ which is not possible

When n = 5, $c=\frac{40}{21}$ which is not possible

When n = 6, $c=\frac{60}{51}$ which is not possible

$\therefore$ we get c = 12 when n = 3

Hence, there exists only one value of c which holds the inequality.

11. **(b)** Putting $\theta=0$, we get $b_0=0$

$\therefore \sin n\theta=\sum_{r=1}^{n} b_r\sin^r\theta \Rightarrow \frac{\sin n\theta}{\sin\theta}=\sum_{r=1}^{n} b_r(\sin\theta)^{r-1}$

$=b_1+b_2\sin\theta+b_3\sin^2\theta+......+b_n\sin^{n-1}\theta$

$\therefore \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta}=b_1\Rightarrow b_1=n.$

12. **(b, c)** For $0<\phi<\frac{\pi}{2}$

$x=\sum_{n=0}^{\infty}\cos^{2n}\phi=1+\cos^2\phi+\cos^4\phi+....\infty$

$\Rightarrow x=\frac{1}{1-\cos^2\phi}=\frac{1}{\sin^2\phi}$ ....(i)

[using sum of infinite G.P., $\cos^2\alpha$ being < 1]

$y=\sum_{n=0}^{\infty}\sin^{2n}\phi=1+\sin^2\phi+\sin^4\phi+....\infty$

$\Rightarrow y=\frac{1}{1-\sin^2\phi}=\frac{1}{\cos^2\phi}$ ....(ii)

and $z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$

$\Rightarrow z=1+\cos^2\phi\sin^2\phi+\cos^4\phi\sin^4\phi+....\infty$

$\Rightarrow z=\frac{1}{1-\cos^2\phi\sin^2\phi}$ ....(iii)

On substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (iii), from (i) and (ii), we get

$z=\frac{1}{1-\frac{1}{x}\cdot\frac{1}{y}}\Rightarrow z=\frac{xy}{xy-1}\Rightarrow xyz-z=xy\Rightarrow xyz=xy+z.$

Now, $x+y+z=\frac{1}{\cos^2\phi}+\frac{1}{\sin^2\phi}+\frac{1}{1-\cos^2\phi\sin^2\phi}$

$=\frac{[\sin^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi]}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}$

$=\frac{(\sin^2\phi+\cos^2\phi)\,(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}$

$=\frac{1}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}=xyz$

$\therefore$ (b) and (c) both are correct.

13. $a_n=\frac{3}{4}-\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+....+(-1)^{n-1}\left(\frac{3}{4}\right)^n$

$=\frac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}}=\frac{3}{7}\left(1-\left(-\frac{3}{4}\right)^n\right)$

$b_n=1-a_n$ and $b_n>a_n\ \forall\ n\geq n_0$

$\therefore$ $1-a_n>a_n \Rightarrow 2a_n<1$

$\Rightarrow \frac{6}{7}\left[1-\left(-\frac{3}{4}\right)^n\right]<1\Rightarrow-\left(-\frac{3}{4}\right)^n<\frac{1}{6}$

$\Rightarrow (-3)^{n+1}<2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for $n\geq 7$.

$\therefore$ The least natural number, for which it holds is 6

[$\because$ it holds for every even natural number]

14. On solving the system of equations, $u+2v+3w=6$, $4u+5v+6w=12$ and $6u+9v=4$ we get $u=-1/3$, $v=2/3$, $w=5/3$

$\therefore$ $u+v+w=2,\ \frac{1}{u}+\frac{1}{v}+\frac{1}{w}=-\frac{9}{10}$

Let r be the common ratio of the G.P., a, b, c, d. Then $b=ar$, $c=ar^2$, $d=ar^3$

Then the first equation

$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+[(b-c)^2+(c-a)^2+(d-b)^2]x+(u+v+w)=0$ becomes

$-\frac{9}{10}x^2+[(ar-ar^2)^2+(ar^2-a)^2+(ar^3-ar)^2]\,x+2=0$

$\Rightarrow 9x^2-10a^2\,(1-r)^2\,[r^2+(r+1)^2+r^2\,(r+1)^2]x-20=0$

$\Rightarrow 9x^2-10a^2\,(1-r)^2\,(r^4+2r^3+3r^2+2r+1)\,x-20=0$

$\Rightarrow 9x^2-10a^2\,(1-r)^2\,(1+r+r^2)^2\,x-20=0,$

$\Rightarrow 9x^2-10a^2\,(1-r^3)^2\,x-20=0$ ....(i)

and $20x^2+10\,(a-d)^2x-9=0$ becomes

$20x^2+10\,(a-ar^3)^2\,x-9=0$

$\Rightarrow 20x^2+10\,a^2\,(1-r^3)^2x-9=0$ ....(ii)

Since Eq. (ii) can be obtained by the substitution $x=1/x$ in Eq. (i), therefore Eqs. (i) and (ii) have reciprocal roots.

15. $S_1=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+....\infty$

$S_2=2+2.\frac{1}{3}+2\left(\frac{1}{3}\right)^2+....\infty$

$S_3=3+3.\frac{1}{4}+3\left(\frac{1}{4}\right)^2+....\infty$

....

....

$S_n=n+n.\frac{1}{n+1}+n\left(\frac{1}{n+1}\right)^2+....\infty$

$\Rightarrow S_1 = \frac{1}{1-\frac{1}{2}} = 2 \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$

$S_2 = \frac{2}{1-\frac{1}{3}} = 3, \qquad S_3 = \frac{3}{1-\frac{1}{4}} = 4,$

$S_n = \frac{n}{1-\frac{1}{n+1}} = (n+1) \qquad \therefore S_1^2 + S_2^2 + S_3^2 + ... + S_{2n-1}^2$

$= 2^2 + 3^2 + 4^2 + ...+ (n+1)^2 + ...+ (2n)^2$

$= [1^2 + 2^2 + 3^2 + 4^2 ..... (2n)^2] - 1$

$= \frac{2n(2n+1)(4n+1)}{6} - 1$

$\left[\because 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{n(2n+1)(4n+1) - 3}{3}$

16. Given : $2 < a, b, c < 18 \qquad a + b + c = 25$ ....(i)

$\because$ 2, a, b are in AP, $\therefore 2a = b + 2$

$\therefore 2a - b = 2$ ....(ii)

$\because$ b, c, 18 are in GP, $\therefore c^2 = 18b$ ....(iii)

From (ii), $a = \frac{b+2}{2}$

From (i), $\frac{b+2}{2} + b + c = 25 \Rightarrow 3b = 48 - 2c$

From (iii), $c^2 = 6(48 - 2c) \Rightarrow c^2 + 12c - 288 = 0$

$\Rightarrow c = 12, -24$ (rejected) $\Rightarrow a = 5, b = 8, c = 12$

17. Consider a G.P. with first term A and common ratio R

$T_p = 27 = AR^{p-1}$ ....(i)

$T_q = 8 = AR^{q-1}$ ....(ii)

$T_r = 12 = AR^{r-1}$ ....(iii)

From (i) and (ii),

$R^{p-q} = \frac{27}{8} \Rightarrow R^{p-q} = (3/2)^3$ ....(iv)

From (ii) and (iii),

$R^{q-r} = \frac{8}{12} \Rightarrow R^{q-r} = (3/2)^{-1}$ ....(v)

From (iv) and (v),

$R = 3/2; p - q = 3; \ q - r = -1$

$p - 2q + r = 4; \quad p, q, r \in N$ ....(vi)

As there can be infinite natural numbers for p, q and r to satisfy equation (vi)

$\therefore$ There can be infinite G.P.'s.

Topic-3: Harmonic Progression, Relation Between A. M., G. M. and H.M. of two Positive Numbers

1. **(d)** $\because a_1, a_2, a_3,$ .... are in H.P.

$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ .... are in A.P.

$\therefore \frac{1}{a_1} = \frac{1}{5}$ and $\frac{1}{a_{20}} = \frac{1}{25}$

$\frac{1}{a_1} + 19d = \frac{1}{a_{20}} \Rightarrow \frac{1}{5} + 19d = \frac{1}{25} \Rightarrow d = \frac{-4}{475}$

Now $\frac{1}{a_n} = \frac{1}{5} + (n-1)\left(\frac{-4}{475}\right)$

Clearly $a_n < 0$, if $\frac{1}{a_n} < 0 \Rightarrow \frac{1}{5} - \frac{4n}{475} + \frac{4}{475} < 0$

$\Rightarrow -4n < -99$ or $n > \frac{99}{4} = 24\frac{3}{4} \quad \therefore n \ge 25$

$\therefore$ Least value of n is 25.

2. **(d)** a, b, c, d are in A.P. $\therefore d, c, b, a$ are also in A.P.

$\Rightarrow \frac{d}{abcd}, \frac{c}{abcd}, \frac{b}{abcd}, \frac{a}{abcd}$ are also in A.P.

$\Rightarrow \frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd}$ are in A.P.

$\therefore$ abc, abd, acd, bcd are in H.P.

3. **(b)** $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{\alpha+\beta}{2\alpha\beta} = \frac{(4+\sqrt{5})}{(5+\sqrt{2})} \cdot \frac{(5+\sqrt{2})}{(8+2\sqrt{5})} \cdot \frac{1}{2} = \frac{1}{4}$

$\therefore H = 4.$

4. **(d)** $a_1 = h_1 = 2, a_{10} = h_{10} = 3$

$3 = a_{10} = 2 + 9d \Rightarrow d = 1/9 \quad \therefore a_4 = 2 + 3d = 7/3$

Now, $3 = h_{10} \Rightarrow \frac{1}{3} = \frac{1}{h_{10}} = \frac{1}{2} + 9D \quad \therefore D = -\frac{1}{54}$

and $\frac{1}{h_7} = \frac{1}{2} + 6D = \frac{1}{2} - \frac{1}{9} = \frac{7}{18} \quad \therefore a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6.$

5. **(d)** Since, $\ell n(a + c)$, $\ell n(a - c)$, $\ell n(a - 2b + c)$ are in A.P.

$\therefore \ell n(a-2b+c) - \ell n(a-c) = \ell n(a-c) - \ell n(a+c)$

$\Rightarrow \ell n\frac{a-2b+c}{a-c} = \ell n\frac{a-c}{a+c} \Rightarrow \frac{a-2b+c}{a-c} = \frac{a-c}{a+c}$

$\Rightarrow a + c, a - c, a - 2b + c$ are in G.P.

$\Rightarrow (c - a)^2 = (a + c)(a - 2b + c)$

$\Rightarrow (c - a)^2 = (a + c)^2 - 2b(a + c)$

$\Rightarrow 2b(a + c) = (a + c)^2 - (c - a)^2$

$\Rightarrow 2b(a + c) = 4ac \Rightarrow b = \frac{2ac}{a+c}$

$\therefore$ a, b, c are in H.P.

6. **(8)** By AM – GM inequality

AM ≥ GM

$\therefore \frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \ge \left[3^{(y_1+y_2+y_3)}\right]^{\frac{1}{3}}$

$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \ge 3^4$

$\Rightarrow \log_3\left(3^{y_1} + 3^{y_2} + 3^{y_3}\right) \ge 4 \Rightarrow m = 4$

$\because \log_3 x_1 + \log_3 x_2 + \log_3 x_3 = \log_3(x_1x_2x_3)$

Again by AM – GM inequality

AM ≥ GM

$\frac{x_1 + x_2 + x_3}{3} \ge \sqrt[3]{x_1x_2x_3} \Rightarrow x_1x_2x_3 \le 27$

$\Rightarrow \log_3(x_1x_2x_3) \le \log_3(3^3)$

$\Rightarrow \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \le 3 \quad \Rightarrow M = 3$

Now, $\log_2(m^3) + \log_3(M^2) = 6 + 2 = 8$

7. Let a and b be two positive numbers.

Their, H.M. $= \frac{2ab}{a+b}$ and $G.M. = \sqrt{ab}$

$\because HM : GM = 4 : 5 \quad \therefore \frac{2ab}{(a+b)\sqrt{ab}} = \frac{4}{5}$

$\Rightarrow \frac{2\sqrt{ab}}{a+b} = \frac{4}{5} \Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$

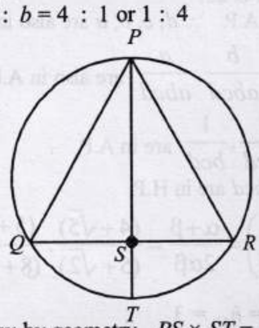
[By Componendo and Dividendo]

$\Rightarrow \dfrac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\dfrac{9}{1} \Rightarrow \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=3,-3$

$\Rightarrow \dfrac{2\sqrt{a}}{2\sqrt{b}}=\dfrac{3+1}{3-1},\dfrac{-3+1}{-3-1} \Rightarrow \dfrac{a}{b}=4,\dfrac{1}{4}$

$\therefore\ a:b=4:1$ or $1:4$

8. **(b, d)**



We know by geometry $PS \times ST = QS \times SR$ ...(i)

$\because$ S is not the centre of circulm circle,

$PS \neq ST$

And we know that for two unequal real numbers.

H.M. < G.M.

$\Rightarrow \dfrac{2}{\frac{1}{PS}+\frac{1}{ST}}<\sqrt{PS\times ST} \Rightarrow \dfrac{1}{PS}+\dfrac{1}{ST}>\dfrac{2}{\sqrt{PS\times ST}}$

$\Rightarrow \dfrac{1}{PS}+\dfrac{1}{ST}>\dfrac{2}{\sqrt{QS\times SR}}$ [using eqn (i)] ...(ii)

$\therefore$ (b) is the correct option.

Also $\sqrt{QS\times SR}<\dfrac{QS+SR}{2}$ ($\because$ GM < AM)

$\Rightarrow \dfrac{1}{\sqrt{QS\times SR}}>\dfrac{2}{QR} \Rightarrow \dfrac{2}{\sqrt{QS\times SR}}>\dfrac{4}{QR}$ ...(iii)

From equations (ii) and (iii), $\dfrac{1}{PS}+\dfrac{1}{ST}>\dfrac{4}{QR}$

$\therefore$ (d) is also the correct option.

9. **(b)** If x, y, z are in G.P. ($x, y, z > 1$); then $\log x$, $\log y$, $\log z$ will be in A.P.

$\Rightarrow$ $1+\log x$, $1+\log y$, $1+\log z$ will also be in A.P.

$\therefore$ $\dfrac{1}{1+\log x},\dfrac{1}{1+\log y},\dfrac{1}{1+\log z}$ will be in H.P.

10. **(a,b,d)** Let x be the first term and y the $(2n-1)$th terms of AP, GP and HP whose nth terms are a, b, c respectively.

For AP, $y = x + (2n-2)\,d$

$\Rightarrow d=\dfrac{y-x}{2(n-1)}$

$\therefore\ a = x+(n-1)\,d = x+\dfrac{1}{2}(y-x)=\dfrac{1}{2}(x+y)$ ....(i)

For G.P., $y = xr^{2n-2} \Rightarrow r=\left(\dfrac{y}{x}\right)^{\frac{1}{2n-2}}$

$\therefore\ b = xr^{n-1} = x.\left(\dfrac{y}{x}\right)^{1/2}=\sqrt{xy}$ ....(ii)

For H.P., $\dfrac{1}{y}=\dfrac{1}{x}+(2n-2)d_1 \Rightarrow d_1=\dfrac{x-y}{2xy\,(n-1)}$

$\dfrac{1}{c}=\dfrac{1}{x}+(n-1)\,d_1=\dfrac{1}{x}+\dfrac{x-y}{2xy}$

$\therefore\ \dfrac{1}{c}=\dfrac{x+y}{2xy} \Rightarrow c=\dfrac{2xy}{x+y}$ ....(iii)

Thus from (i), (ii) and (iii), a, b, c are A.M., G.M. and H.M. respectively of x and y.

11. **(c)** Given $A_1=\dfrac{a+b}{2}, G_1=\sqrt{ab},\ H_1=\dfrac{2ab}{a+b}$

Also $A_n=\dfrac{A_{n-1}+H_{n-1}}{2}, G_n=\sqrt{A_{n-1}H_{n-1}}$

$H_n=\dfrac{2A_{n-1}H_{n-1}}{A_{n-1}+H_{n-1}}$

$\Rightarrow G_n^2=A_nH_n \Rightarrow A_nH_n=A_{n-1}H_{n-1}$

Similarly we can prove

$A_nH_n=A_{n-1}H_{n-1}=A_{n-2}H_{n-2}=....=A_1H_1$

$\Rightarrow A_nH_n=ab$

$\therefore\ G_1^2=G_2^2=G_3^2 ....=ab$

$\Rightarrow G_1=G_2=G_3 ....=\sqrt{ab}$

12. **(a)** $A_n=\dfrac{A_{n-1}+H_{n-1}}{2}$

$\therefore\ A_n-A_{n-1}=\dfrac{A_{n-1}+H_{n-1}}{2}-A_{n-1}$

$=\dfrac{H_{n-1}-A_{n-1}}{2}<0\ (\because A_{n-1}>H_{n-1})$

$\Rightarrow A_n<A_{n-1}$ or $A_{n-1}>A_n$

$\therefore\ A_1>A_2>A_3>....$

13. **(b)** $A_nH_n=ab \Rightarrow H_n=\dfrac{ab}{A_n}$

$\because\ \dfrac{1}{A_{n-1}}<\dfrac{1}{A_n} \Rightarrow H_{n-1}<H_n \quad \therefore H_1<H_2<H_3<.......$

14. **(c)** Given : a_1, a_2, a_3, a_4 are in G.P.

Then b_1, b_2, b_3, b_4 are the numbers

$a_1, a_1+a_2, a_1+a_2+a_3,\ a_1+a_2+a_3+a_4$

or $a, a+ar, a+ar+ar^2, a+ar+ar^2+ar^3$

Since, above numbers are neither in A.P. nor in G.P. Therefore, statement 1 is true.

Also $\dfrac{1}{a},\dfrac{1}{a+ar},\dfrac{1}{a+ar+ar^2},\dfrac{1}{a+ar+ar^2+ar^3}$ are not in A.P. $\therefore b_1, b_2, b_3, b_4$ are not in H.P.

$\therefore$ Statement 2 is false.

15. Given : a, b, c are in A.P.

$\therefore\ 2b=a+c$ ....(i)

Now a^2, b^2, c^2 are in H.P.

$\Rightarrow \dfrac{1}{b^2}-\dfrac{1}{a^2}=\dfrac{1}{c^2}-\dfrac{1}{b^2}$

$\Rightarrow \dfrac{(a-b)(a+b)}{b^2a^2}=\dfrac{(b-c)(b+c)}{b^2c^2}$

$\Rightarrow ac^2+bc^2=a^2b+a^2c$ ($\because a-b=b-c$)

$\Rightarrow ac\,(c-a)+b\,(c-a)\,(c+a)=0$

$\Rightarrow (c-a)\,(ab+bc+ca)=0$

$\Rightarrow$ Either $c-a=0$ or $ab+bc+ca=0$

$\Rightarrow$ Either $c=a$ or $(a+c)\,b+ca=0$

and hence from (i), $2b^2+ca=0$

Either $a=b=c$ or $b^2=a\left(\dfrac{-c}{2}\right)$

$\therefore$ a, b, – c/2 are in G.P.

16. Clearly $A_1+A_2=a+b$

$\dfrac{1}{H_1}+\dfrac{1}{H_2}=\dfrac{1}{a}+\dfrac{1}{b}$

$$\Rightarrow \frac{H_1+H_2}{H_1H_2}=\frac{a+b}{ab}=\frac{A_1+A_2}{G_1G_2}$$

$$\Rightarrow \frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$$

Also $\frac{1}{H_1}=\frac{1}{a}+\frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right) \Rightarrow H_1=\frac{3ab}{2b+a}$

$$\frac{1}{H_2}=\frac{1}{a}+\frac{2}{3}\left(\frac{1}{b}-\frac{1}{a}\right) \Rightarrow H_2=\frac{3ab}{2a+b}$$

$$\Rightarrow \frac{A_1+A_2}{H_1+H_2}=\frac{a+b}{3ab\left(\frac{1}{2b+a}+\frac{1}{2a+b}\right)}$$

$$=\frac{(2b+a)(2a+b)}{9ab}$$

17. Given that $a_1, a_2, \ldots, a_n$ are positive real numbers in G.P.

$\therefore a_1=a,\ a_2=ar,\ a_3=ar^2,\ \vdots,\ a_n=ar^{n-1}$ — As $a_1, a_2, \ldots, a_n$ are positive $\therefore r>0$

A_n is A.M. of $a_1, a_2, \ldots, a_n$

$$\therefore A_n=\frac{a_1+a_2+\ldots+a_n}{n}=\frac{a+ar+\ldots+ar^{n-1}}{n}$$

$$A_n=\frac{a(1-r^n)}{n(1-r)} \quad \ldots\text{(i)} \quad (r\neq 1)$$

G_n is G.M. of $a_1, a_2, \ldots, a_n$

$$\therefore G_n=\sqrt[n]{a_1, a_2 \ldots a_n}=\sqrt[n]{a.ar.ar^2\ldots ar^{n-1}}$$

$$=\sqrt[n]{a^n, r^{\frac{n(n-1)}{2}}}$$

$$G_n=ar^{\frac{n-1}{2}} \quad \ldots\text{(ii)} \quad (r\neq 1)$$

$\because H_n$ is H.M. of $a_1, a_2, \ldots, a_n$

$$\therefore H_n=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}=\frac{n}{\frac{1}{a}+\frac{1}{ar}+\ldots+\frac{1}{ar^{n-1}}}$$

$$=\frac{n}{\frac{1}{a}\frac{\left(\frac{1}{r^n}-1\right)}{\left(\frac{1}{r}-1\right)}}=\frac{n}{\frac{1}{a}\left(\frac{1-r^n}{r^n}\right)\cdot\frac{r}{1-r}}$$

$$H_n=\frac{anr^{n-1}(1-r)}{(1-r^n)} \quad \ldots\text{(iii)} \quad (r\neq 1)$$

We also observe that

$$A_nH_n=\frac{a(1-r^n)}{(n(1-r))}\times\frac{anr^{n-1}(1-r)}{(1-r^n)}=a^n r^{n-1}=G_n^2$$

$$\therefore A_nH_n=G_n^2 \quad \ldots\text{(iv)}$$

$\therefore$ Now, G.M. of $G_1, G_2, \ldots G_n$ is

$$G=\sqrt[n]{G_1\, G_2 \ldots G_n}$$

$$G=\sqrt[n]{\sqrt{A_1H_1}\ \sqrt{A_2H_2}\ldots\sqrt{A_nH_n}}$$

[using equation (iv)]

$$G=(A_1A_2\ldots A_nH_1H_2\ldots H_n)^{1/2n} \quad \ldots\text{(v)}$$

If $r=1$, then

$A_n=G_n=H_n=a$

Also $A_n\,H_n=G_n^2$

$\therefore$ For $r=1$ also, equation (v) holds.

$\therefore G=(A_1A_2\ldots A_nH_1H_2\ldots H_n)^{1/2n}$

18. Let a and b be two numbers and $A_1, A_2, A_3, \ldots A_n$ be n A.M's between a and b.

$\therefore$ a, $A_1, A_2, \ldots, A_n$, b are in A.P.

Since, there are $(n+2)$ terms in the series, therefore

$$a+(n+1)d=b \Rightarrow d=\frac{b-a}{n+1}$$

$$\therefore A_1=a+\frac{b-a}{n+1}=\frac{an+b}{n+1}$$

$$\therefore p=\frac{an+b}{n+1} \quad \ldots\text{(i)}$$

The first H.M. between a and b, when n HM's are inserted between a and b can be obtained by replacing a by $\frac{1}{a}$ and b by $\frac{1}{b}$ in Eq. (i) and then taking its reciprocal.

$$\therefore q=\frac{1}{\frac{\left(\frac{1}{a}\right)n+\frac{1}{b}}{n+1}}=\frac{(n+1)ab}{bn+a}$$

$$\therefore q=\frac{(n+1)ab}{a+bn} \quad \ldots\text{(ii)}$$

Now, we have to prove that q cannot lie between p and $\frac{(n+1)^2}{(n-1)^2}p$.

Now, $n+1>n-1 \Rightarrow \frac{n+1}{n-1}>1$

$$\Rightarrow \left(\frac{n+1}{n-1}\right)^2>1 \text{ or } p\left(\frac{n+1}{n-1}\right)^2>p$$

$$\Rightarrow p<p\left(\frac{n+1}{n-1}\right)^2 \quad \ldots\text{(iii)}$$

If q does not lie between p and q, then q is either less than p or $q>p\left(\frac{n+1}{n-1}\right)^2$.

Now, $\frac{p}{q}=\frac{(na+b)\,(nb+a)}{(n+1)^2\,ab}$

$$\Rightarrow \frac{p}{q}-1=\frac{n(a^2+b^2)+ab(n^2+1)-(n+1)^2\,ab}{(n+1)^2\,ab}$$

$$\Rightarrow \frac{p}{q}-1=\frac{n(a^2+b^2-2ab)}{(n+1)^2\,ab}$$

$$\Rightarrow \frac{p}{q}-1=\frac{n}{(n+1)^2}\left(\frac{a-b}{\sqrt{ab}}\right)^2=\frac{n}{(n+1)^2}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2$$

$$\Rightarrow \frac{p}{q}-1>0$$

$$\therefore p>q \quad \ldots\text{(iv)}$$

From (iii) and (iv), we get, $q < p < \left(\frac{n+1}{n-1}\right)^2 p$

$\therefore$ q can not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

19. Given that $a, b, c > 0$

We know for positive numbers, A.M. ≥ G.M.

$\therefore$ For positive numbers a, b, c, we get

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad ....(i)$$

Also for positive numbers $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$;

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{abc}} \quad ....(ii)$$

On multiplying eqs (i) and (ii), we get

$$\left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \geq \sqrt[3]{abc} \times \frac{1}{\sqrt[3]{abc}}$$

$$\therefore \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

20. Let the two numbers be a and b, then

$$\frac{a+b}{2} = A; \sqrt{ab} = G; \frac{2ab}{a+b} = 4 \quad ....(i)$$

Now, $2A + G^2 = 27 \Rightarrow a + b + ab = 27$ ....(ii)

On putting $ab = 27 - (a+b)$ in Eq. (i),

$$\frac{54-2(a+b)}{a+b} = 4 \Rightarrow a+b = 9, \text{ then } ab = 27 - 9 = 18$$

On solving (i) and (ii), we get $a = 6, b = 3$ or $a = 3, b = 6$, which are the required numbers.

Topic-4: Arithmetic-Geometric Sequence (A.G.S.), Some Special Sequences

1. **(1219)** Given $S = 77 + 757 + 7557 + ... + \overbrace{75...57}^{98}$

$$10S = \quad 770 + 7570 + ... + 75.....570 + \overbrace{75...570}^{98}$$

$$-9S = 77 - \underbrace{13 - 13 .... - 13}_{98\text{ times}} - \underbrace{75 .... 570}_{98}$$

$$9S = -77 + 13 \times 98 + \underbrace{75.....57}_{99} + 13$$

$$S = \frac{\overbrace{75.......57}^{99} + 1210}{9}$$

$\Rightarrow$ m = 1210 and n = 9 $\Rightarrow$ m + n = 1219.

2. Since n is an odd integer, $(-1)^{n-1} = 1$ and $n-1, n-3, n-5, ...$ are even integers.

Now, $n^3 - (n-1)^3 + (n-2)^3 - (n-3)^3 + ... + (-1)^{n-1}1^3$

$= n^3 + (n-1)^3 + (n-2)^3 + ... + 1^3 - 2\,[(n-1)^3 + (n-3)^3 + ... + 2^3]$

$= [n^3 + (n-1)^3 + (n-2)^3 + ... + 1^3]$

$$-2 \times 2^3 \left[\left(\frac{n-1}{2}\right)^3 + \left(\frac{n-3}{2}\right)^3 + ... + 1^3\right] \quad .... (i)$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - 16\left[\left\{\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right\}\right]^2$$

[Since in the equation (i), the first square bracket contain the sum of cubes of 1st n natural numbers. Whereas the second square bracket contains the sum of the cubes of natural numbers from 1 to $\left(\frac{n-1}{2}\right)$ as $(n-1), (n-3), ...$ are even integers.]

$$= \frac{1}{4}n^2(n+1)^2 - 16\,\frac{(n-1)^2(n+1)^2}{16 \times 4}$$

$$= \frac{1}{4}(n+1)^2[n^2 - (n-1)^2] = \frac{1}{4}(n+1)^2(2n-1)$$

3. For n to be n is odd, let $n = 2m + 1$

$\therefore$ Required sum

$= 1^2 + 2.2^2 + 3^2 + 2.4^2 + ... + 2\,(2m)^2 + (2m+1)^2$

$= \Sigma(2m+1)^2 + 4[1^2 + 2^2 + 3^2 + ... + m^2]$

$$= \frac{(2m+1)(2m+2)(4m+2+1)}{6} + \frac{4m(m+1)(2m+1)}{6}$$

$$= \frac{(2m+1)(m+1)}{6}[2(4m+3) + 4m]$$

$$= \frac{(2m+1)(2m+2)(6m+3)}{6} = \frac{(2m+1)^2(2m+2)}{2}$$

$$= \frac{n^2(n+1)}{2} \quad [\because 2m+1 = n)$$

4. **(a, b, c)** Since α, β are roots of $x^2 - x - 1 = 0$ with $\alpha > \beta$

$$\therefore \quad \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

Also, $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha - \beta = \sqrt{5}$

$\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1;\ b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, n \geq 2$$

Let us now check the given options, one by one

(a) $$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\sqrt{5}(10^n)} = \frac{1}{\sqrt{5}}\left[\sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}\right] = \frac{1}{\sqrt{5}}\left[\frac{\alpha}{10-\alpha} - \frac{\beta}{10-\beta}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{10\alpha - \alpha\beta - 10\beta + \alpha\beta}{(10-\alpha)(10-\beta)}\right]$$

$$= \frac{1}{5}\left[\frac{10(\alpha-\beta)}{100 - 10(\alpha+\beta) + \alpha\beta}\right] = \frac{1}{5}\left[\frac{10\sqrt{5}}{100-10-1}\right] = \frac{10}{89}$$

Thus option (a) is correct.

(b) $$b_n = a_{n+1} + a_{n-1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

$$= \frac{(\alpha^{n+1} + \alpha^{n-1}) - (\beta^{n+1} + \beta^{n-1})}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1}(\alpha^2+1) - \beta^{n-1}(\beta^2+1)}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}[\alpha^{n-1}(\alpha+2) - \beta^{n-1}(\beta+2)]$$

[using $\alpha^2 = \alpha + 1, \beta^2 = \beta + 1$]

$$= \frac{1}{\sqrt{5}}\left[\alpha^{n-1}\left(\frac{1+\sqrt{5}}{2}+2\right) - \beta^{n-1}\left(\frac{1-\sqrt{5}}{2}+2\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{n-1}\left(\frac{5+\sqrt{5}}{2}\right) - \beta^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)\right]$$

$$= \frac{\sqrt{5}}{\sqrt{5}}\left[\alpha^{n-1}\left(\frac{\sqrt{5}+1}{2}\right)+\beta^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)\right]$$

$= \alpha^{n-1}\alpha + \beta^{n-1}\beta = \alpha^n + \beta^n$

Thus option(b) is correct.

(c) $a_1 + a_2 + a_3 + ... + a_n$

$$= \frac{\alpha^1-\beta^1}{\alpha-\beta}+\frac{\alpha^2-\beta^2}{\alpha-\beta}+\frac{\alpha^3-\beta^3}{\alpha-\beta}+....+\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$= \frac{1}{\sqrt{5}}[(\alpha+\alpha^2+\alpha^3+.....+\alpha^n)-(\beta+\beta^2+\beta^3+.....+\beta^n)]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\alpha(1-\alpha^n)}{1-\alpha}-\frac{\beta(1-\beta^n)}{1-\beta}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\alpha(1-\beta)(1-\alpha^n)-\beta(1-\alpha)(1-\beta^n)}{(1-\alpha)(1-\beta)}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(\alpha-\alpha\beta)(1-\alpha^n)-(\beta-\alpha\beta)(1-\beta^n)}{(1-\alpha)(1-\beta)}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(\alpha+1)(1-\alpha^n)-(\beta+1)(1-\beta^n)}{1-(\alpha+\beta)+\alpha\beta}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\alpha^2(1-\alpha^n)-\beta^2(1-\beta^n)}{1-1-1}\right]$$

[using $\alpha^2-\alpha-1=0, \beta^2-\beta-1=0$]

$$= \frac{1}{\sqrt{5}}\left[\frac{(\alpha^2-\beta^2)-(\alpha^{n+2}-\beta^{n+2})}{-1}\right]$$

$$= \left[\frac{-(\alpha-\beta)(\alpha+\beta)}{\alpha-\beta}+\frac{(\alpha^{n+2}-\beta^{n+2})}{\alpha-\beta}\right] = -1 + a_{n+2}$$

Thus option (c) is correct.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{b_n}{10^n}=\sum_{n=1}^{\infty}\frac{\alpha^n+\beta^n}{10^n}$ [using $b_n = \alpha^n + \beta^n$ from (b)]

$$= \sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n+\left(\frac{\beta}{10}\right)^n = \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}}+\frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}$$

$$= \frac{\alpha}{10-\alpha}+\frac{\beta}{10-\beta} = \frac{\alpha(10-\beta)+\beta(10-\alpha)}{(10-\alpha)(10-\beta)}$$

$$= \frac{10(\alpha+\beta)-2\alpha\beta}{100-10(\alpha+\beta)+\alpha\beta} = \frac{10+2}{100-10-1} = \frac{12}{89}$$

Thus option (d) is incorrect.

5. (a, d) We have

$$a(n) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+.....+\frac{1}{2^n-1}$$

$$= \left[1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+.....+\frac{1}{15}\right)+...+\left(\frac{1}{2^{n-1}}+.....+\frac{1}{2^n-1}\right)\right]$$

$$< \left[1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+.....+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+...+\frac{1}{8}\right)+..+\left(\frac{1}{2^{n-1}}+..+\frac{1}{2^{n-1}}\right)\right]$$

$$< \left[1+\frac{2}{2}+\frac{4}{4}+\frac{8}{8}+.....+\frac{2^{n-1}}{2^{n-1}}\right] = [1+1+.....+1] = n$$

$\therefore\ a(100) < 100$

Also

$$a(n) = \left[1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+.....+\left(\frac{1}{2^{n-1}+1}+..+\frac{1}{2^n}\right)-\frac{1}{2^n}\right]$$

$$> \left[1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+...+\frac{1}{8}\right)+...+\left(\frac{1}{2^n}+..+\frac{1}{2^n}\right)-\frac{1}{2^n}\right]$$

$$= 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+...+\frac{2^{n-1}}{2^n}-\frac{1}{2^n}$$

$$= 1+\underbrace{\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....+\frac{1}{2}\right)}_{n\text{ times}}-\frac{1}{2^n}$$

$$= 1+\frac{n}{2}-\frac{1}{2^n} = \left(1-\frac{1}{2^n}\right)+\frac{n}{2}$$

$$\therefore\ a(200) > \left[\left(1-\frac{1}{2^{200}}\right)+\frac{200}{2}\right] > 100,$$

$\therefore\ a(200) > 100.$

6. Given series

$$= \sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+.... \text{ up to } m \text{ terms}\right]$$

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+.... \text{to } m \text{ terms}\right]$$

Now,

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r = 1-{}^nC_1.\frac{1}{2}+{}^nC_2.\frac{1}{2^2}-{}^nC_3.\frac{1}{2^3}+...$$

$$= \left(1-\frac{1}{2}\right)^n = \frac{1}{2^n}$$

Similarly, $\displaystyle\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r = \left(1-\frac{3}{4}\right)^n = \frac{1}{4^n}$ etc.

Hence the sum of the series

$$= \frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{16^n}+.... \text{to } m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left(1-\left(\frac{1}{2^n}\right)^m\right)}{1-\frac{1}{2^n}} \quad \text{[Sum of G.P.]}$$

$$= \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

7. Given : $n = p_1^{\alpha_1}.\ p_2^{\alpha_2}.\ p_3^{\alpha_3}\ ....\ p_k^{\alpha_k}$ ....(i)

Where $n \in N$ and $p_1, p_2, p_3, ..., p_k$ are distinct prime numbers.

Taking log on both sides of Eq. (i), we get

$\log n = \alpha_1 \log p_1 + \alpha_2 \log p_2 + ...+ \alpha_k \log p_k$ ....(ii)

Since $p_1, p_2, p_3, ..., p_k$ are distinct prime numbers, therefore maximum any one prime number out of $p_1, p_2, p_3, ..., p_k$ will be 2 and the remaining prime numbers will be greater than 2.

$\therefore$ $\log n \geq \alpha_1 \log 2 + \alpha_2 \log 2 + \alpha_3 \log 2 + ...+ \alpha_k \log 2$

$\Rightarrow$ $\log n \geq (\alpha_1+\alpha_2+ ....+ \alpha_k) \log 2$

$\Rightarrow$ $\log n \geq k \log 2$